Long-range-interaction models and a deformed-loop symmetry

Mo-Lin Ge and Yiwen Wang
Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China

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The generalized Sutherland-Römer and Yan models with an internal spin degree of freedom are formulated in terms of both the Polychronakos' approach [Phys. Rev. Lett. **69**, 703 (1992)] and the RTT relation [see *Integrable Quantum Field Theories,* edited by J. Hietarinta and C. Montonen, Lecture Notes in Physics Vol. 151 (Springer, New York, 1982), pp. 61–119] associated with the Yang-Baxter equation in a consistent way. A deformed-loop symmetry is shown to generate both of the models. We finally introduce the reflection algebra $K(u)$ to long-range-interaction models.

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I. INTRODUCTION

In the last few years, a number of one-dimensional long-range-interaction models have been studied [1–10]. A typical one is the Calogero-Sutherland (CS) model [1,2] which is subsequently extended to the models with internal spin degrees of freedom [5–9]. Among them there is an interesting approach that was proposed by Bernard, Gaudin, Haldane, and Pasquier (BGHP), who related this type of model to the RTT relation [17] associated with the Yang-Baxter equation YBE [10]. The BGHP approach provides a method to deal with long-range-interaction models: for a given rational solution of YBE, for example, $R(u)=u+P$, where P is the permutation and u the spectral parameter and the RTT relation gives rise to the Yangian (deformed-loop) symmetry. With a particular realization of this symmetry, in general, we can generate corresponding Hamiltonians of the considered systems.

On the other hand, Polychronakos had formulated the integrability in terms of the "coupled" momentum operators [5,6]:

$$\pi_i = p_i + i \sum_{j\neq i} V_{ij} K_{ij} \ , \tag{1.1}$$

where $p_i=-i(\partial/\partial x_i)$ ($\hbar=1$), and $V_{ij}=V(x_i-x_j)$ a potential to be determined, and K_{ij} are the particle permutation operators. The requirements for the Hermiticity of π_i, the absence of linear terms in p_i, and that only the two-body potentials in the Hamiltonian lead to [5]

$$\begin{aligned} &V(x)=-V(-x) \ , \\ &H_0 \equiv \tfrac{1}{2}\sum_i \pi_i^2 = \sum_i p_i^2 + \tfrac{1}{2}\sum_{i\neq j}\left[\frac{\partial}{\partial x_i} V_{ij}K_{ij} + V_{ij}^2\right] \\ &\qquad -\tfrac{1}{6}\sum_{i\neq j\neq k\neq i} V_{ijk}K_{ijk} \ , \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} &V_{ijk}=V_{ij}V_{jk}+V_{jk}V_{ki}+V_{ki}V_{ij}=W_{ij}+W_{jk}+W_{ki} \ , \\ &K_{ijk}=K_{ij}K_{jk} \ , \end{aligned} \tag{1.3}$$

with $W_{ij}=W(x_i-x_j)$ being a symmetric function. The commutation relation between π_i and π_j is found to be

$$[\pi_i,\pi_j] = \sum_{k\neq i,j} V_{ijk}(K_{ijk}-K_{jik}) \ . \tag{1.4}$$

This approach can be applied to many integrable systems, especially to the CS model [5,6].

Recently, Sutherland and Römer (SR) presented a new long-range-interaction model with the Hamiltonian [11]

$$H_{\rm SR} = \tfrac{1}{2}\sum_i p_i^2 + \sum_{i<j} l(l-1)\left[\frac{P_{ij}^+}{sh^2 x_{ij}} - \frac{P_{ij}^-}{\cosh^2 x_{ij}}\right] \ , \tag{1.5}$$

where

$$x_{ij}=x_i-x_j, \quad P_{ij}^{\pm}=\frac{1\pm\sigma_i\sigma_j}{2} \quad (\sigma_i^2=1) \tag{1.6}$$

and a, l are arbitrary parameters. Sutherland and Römer had proved that Eq. (1.5) is quantum integrable. Parallel to this development Yan proposed another model [12]:

$$H_Y = \tfrac{1}{2}\sum_i p_i^2 + \tfrac{1}{2}\sum_{i\neq j} l\delta(x_i-x_j)P_{ij}^+ \tag{1.7}$$

that was solved in terms of the Bethe ansatz. So far both the SR model and the Yan model have not systematically been studied in terms of the RTT relation.

In this paper we shall show the following points: (i) The models, Eqs. (1.5) and (1.7), are also the conclusion of Polychronakos' approach. (ii) On the basis of the RTT relation, the models, Eqs. (1.5) and (1.7), are related to the realization of the Yangian, namely, they belong to the Yang-Baxter system. Both (i) and (ii) are consistent with each other. (iii) Further properties have been discussed that lead to other complicated conserved quantities.

II. SUTHERLAND-RÖMER AND YAN MODELS

Let us first discuss the extended forms of V_{ij} in Eq. (1.1) that are different from those given by Refs. [5,6]. Setting

$$V_{ij} = P_{ij}^+ a_{ij} + P_{ij}^- b_{ij} \ , \tag{2.1}$$

where $P_{ij}^{\pm}$ are given by Eq. (1.6) and σ_i quantum operators obeying

$$\sigma_i K_{ij} = K_{ij}\sigma_j, \quad \sigma_i K_{mn} = K_{mn}\sigma_i \quad (i\neq m,n) \ ,$$

then by substituting Eq. (2.1) into Eq. (1.1) and doing the parallel discussion in Ref. [5], we find

$$V_{ijk}=P^{+}_{ijk}A_{ijk}+P^{-}_{ijk}A_{ijk}+P^{-}_{kij}A_{kij}+P^{-}_{jki}A_{jki}\ , \quad (2.2)$$

where

$$P^{\pm}_{ijk}=P^{\pm}_{ij}P^{\pm}_{ik}\ ,$$
$$A_{ijk}=a_{ij}a_{jk}+a_{jk}a_{ki}+a_{ki}a_{ij}\ ,$$
$$B_{ijk}=a_{ij}b_{jk}+b_{jk}a_{ki}+b_{ki}a_{ij}\ .$$

Note that $P^{+}_{ijk}=P^{+}_{ikj}=\cdots=P^{+}_{kji}$, but $P^{-}_{ijk}=P^{-}_{jik}$ only. The sufficient condition of the quantum integrability of Eq. (1.1) is [5,6]

$$V_{ijk}=\text{const}\ \ (\text{or zero})\ . \quad (2.3)$$

Now let us look for a new solution to Eq. (2.3).

(1) When $A_{ijk}\neq 0$, $B_{ijk}\neq 0$, a sufficient solution can be checked:

$$\begin{aligned} a(x)&=l\coth(ax)\ \ [\text{or } a(x)=l\cot(ax)]\ ,\\ b(x)&=l\tanh(ax)\ \ [\text{or } b(x)=l\tan(ax)]\ , \end{aligned} \quad (2.4)$$

where $x\equiv x_{ij}=x_i-x_j$, and a and l are constants, and

$$V_{ijk}=-l^2(P^{+}_{ijk}+P^{-}_{ijk}+P^{-}_{kij}+P^{-}_{jki})=-l^2\ . \quad (2.5)$$

Define [6]

$$H=\tfrac{1}{2}\sum_i \pi_i^2-\frac{l^2}{6}\sum_{i\neq j\neq k\neq i}K_{ijk}\ ; \quad (2.6)$$

then Eq. (2.5) leads to

$$H=\tfrac{1}{2}\sum_i p_i^2+\sum_{i<j}l(l-aK_{ij})\left[\frac{P^{+}_{ij}}{\sinh^2(ax_{ij})}-\frac{P^{-}_{ij}}{\cosh^2(ax_{ij})}\right]. \quad (2.7)$$

Equation (2.7) is exactly $H_{\rm SR}$ given by SR [11] when $K_{ij}=\pm 1$.

Define

$$\bar{\pi}_i=\pi_i+il\sum_{i\neq j}K_{ij}\ ; \quad (2.8)$$

then

$$[\bar{\pi}_i,\bar{\pi}_j]=2il(\bar{\pi}_i-\bar{\pi}_j)K_{ij}\ , \quad (2.9)$$

$$[H,\pi_i]=[H,\bar{\pi}_i]=0\ . \quad (2.10)$$

The conserved quantities are given by

$$I_n=\sum_i \bar{\pi}_i^n\ , \quad (2.11)$$

which leads to

$$[I_n,I_m]=0\ , \quad (2.12)$$

$$[H,I_n]=0\ , \quad (2.13)$$

i.e., the model is quantum integrable according to Polychronakos [5,6].

(2) When $B_{ijk}=0$, we consider two cases.

(a) $A_{ijk}=0$:

$$\begin{aligned} &a(x)=\frac{l}{x},\quad V_{ijk}=0\ ,\\ &H=\tfrac{1}{2}\sum_i p_i^2+\tfrac{1}{2}\sum_{i\neq j}\frac{l(l-K_{ij})}{(x_i-x_j)^2}P^{+}_{ij}\ , \end{aligned} \quad (2.14)$$

which is well known as the Calogero model when P^{+}_{ij} takes the value 1.

(b) $A_{ijk}=\beta^2\neq 0$:

$$[\pi_i,\pi_j]=\beta\sum_{k\neq i,i}P^{+}_{ijk}(K_{ijk}-K_{jik})\ . \quad (2.15)$$

Define

$$\bar{\pi}_i=\pi_i+\beta\sum_{i\neq j}P^{+}_{ij}K_{ij}\ ; \quad (2.16)$$

it is easy to prove that

$$[\bar{\pi}_i,P^{+}_{jk}]=0,\quad \forall i\ \text{ and } j\neq k\ , \quad (2.17)$$

and

$$[\bar{\pi}_i,\bar{\pi}_j]=2\beta P^{+}_{ij}(\bar{\pi}_i-\bar{\pi}_j)K_{ij}\ , \quad (2.18)$$

$$[\bar{\pi}_i^n,\bar{\pi}_j]=2\beta P^{+}_{ij}(\bar{\pi}_i^n-\bar{\pi}_j^n)K_{ij}\ , \quad (2.19)$$

so that Eq. (2.12) is also satisfied. Define

$$H=\tfrac{1}{2}\sum_i \pi_i^2+\frac{\beta^2}{6}\sum_{i\neq j\neq k\neq i}P^{+}_{ijk}K_{ijk}\ . \quad (2.20)$$

With the help of Eq. (2.15), one can prove

$$[H,\pi_i]=[H,\bar{\pi}_i]=[H,I_n]=0\ . \quad (2.21)$$

For case (b) we have two sufficient solutions of V_{ijk}.

(i)

$$\begin{aligned} &a(x)=il\cot(ax)\ \ [\text{or } a(x)=l\coth(ax)]\ ,\\ &V_{ijk}=-l^2P^{+}_{ijk}\ ,\\ &H=\tfrac{1}{2}\sum_i p_i^2+\tfrac{1}{2}\sum_{i\neq j}\frac{l(l-aK_{ij})}{\sin^2[a(x_i-x_j)]}P^{+}_{ij}\ . \end{aligned} \quad (2.22)$$

Equation (2.22) is the generalization of the spin chain model considered by BGHP [10].

(ii)

$$\begin{aligned} &a(x)=l\,\mathrm{sgn}(x)\ ,\\ &H=\tfrac{1}{2}\sum_i p_i^2+\tfrac{1}{2}\sum_{i\neq j}l(l-K_{ij})\delta(x_i-x_j)P^{+}_{ij}\ . \end{aligned} \quad (2.23)$$

With the condition that $K_{ij}=\pm 1$, Eq. (2.23) was first pointed out by Yan [12] through the Bethe ansatz; he also found the Y operator defined by Yang [13,14] for Eq. (2.23):

$$\begin{aligned} Y^{\alpha\beta}_{ij}=&\frac{1}{ik_{ij}(ik_{ij}-2c)}[ik_{ij}-c(1-\sigma_i\sigma_j)]\\ &\times[-ik_{ij}P^{\alpha\beta}+c(1+\sigma_i\sigma_j)]\ , \end{aligned} \quad (2.24)$$

where P is the permutation, $\sigma_i^2=1$, Y satisfies [13]

$$Y^{\alpha\beta}_{jk}Y^{\beta\gamma}_{ik}Y^{\alpha\beta}_{ij}=Y^{\beta\gamma}_{ij}Y^{\alpha\beta}_{ik}Y^{\beta\gamma}_{jk}\ , \quad (2.25)$$

and $c=l(l\pm 1)/2$ for $K_{ij}=\pm 1$. Note that there is only P_{ij}^{+} in the Hamiltonian equation (2.23) for the quantum integrability.

In this section we have reinterpreted the models, Eqs. (1.5) and (1.7), from the point of view of the formulation Eq. (1.1). Next we shall set up the Yangian [15] description of the models, Eqs. (1.5) and (1.7), through the RTT relation.

III. RTT RELATION AND LONG-RANGE-INTERACTION MODELS

Let us apply the BGHP approach [10] to the SR model and Yan model. The solution of the Yang-Baxter equation, R matrix, takes the simplest form as

$$R(u)=u+\lambda P_{00'} , \tag{3.1}$$

and the RTT relation reads

$$R_{00'}(u-v)T^{0}(u)T^{0'}(v)=T^{0'}(v)T^{0}(u)R_{00'}(u-v) , \tag{3.2}$$

where $T^{0}(u)=T(u)\otimes 1$, $T^{0'}=1\otimes T(u)$, and $P_{00'}$ is the permutation operator exchanging the two auxiliary spaces 0 and 0'. Then we make the expansion [10]

$$T^{0}(u)=I+\sum_{a,b=1}^{p} X_{ba}^{0}\sum_{n=0}^{\infty}\lambda T_{n}^{ab}/u^{(n+1)} , \tag{3.3}$$

$$P_{00'}=\sum_{a,b=1}^{p} X_{ba}^{0}X_{ab}^{0'} . \tag{3.4}$$

It is well known that $\{T_n^{ab}\}$ generate the Yangian [15]. Substituting Eqs. (3.1), (3.3), and (3.4) into Eq. (3.2), one finds

$$\sum_{a,b}\sum_{cd} X_{ba}^{0}X_{dc}^{0'}\sum_{n=0}^{\infty}\left\{u^{-n-1}f_1^{n}-v^{-n-1}f_2^{n}+\sum_{m=0}^{\infty}u^{-n-1}v^{-m-1}f_3^{n,m}\right\}=0 , \tag{3.5}$$

where

$$f_1^{n}=\delta_{bc}T_n^{ad}-\delta_{ad}T_n^{cb}-[T_n^{ab},T_0^{cd}] ,$$

$$f_2^{n}=\delta_{bc}T_n^{ad}-\delta_{ad}T_n^{cb}-[T_0^{ab},T_n^{cd}] ,$$

$$f_3^{n,m}=\lambda(T_n^{ad}T_m^{cb}-T_m^{ad}T_n^{cb})+[T_{n+1}^{ab},T_m^{cd}]-[T_n^{ab},T_{m+1}^{cd}] .$$

For any auxiliary space $\{X_{ab}\}$, we require $f_1^{n}=f_2^{n}=f_3^{n,m}=0$. Obviously, $f_1^{n}=0$ is equivalent to $f_2^{n}=0$. So we need only take

$$f_1^{n}\equiv f_3^{n,m}=0 \tag{3.6}$$

into account.

First, from $f_3^{n,0}=0$ it follows that

$$\delta_{bc}T_{n+1}^{ad}-\delta_{ad}T_{n+1}^{cb}=\lambda(T_0^{ad}T_n^{cb}-T_n^{ad}T_0^{cb})+[T_n^{ab},T_1^{cd}] , \tag{3.7}$$

which can be recast into

$$T_{n+1}^{ad}=\lambda(T_0^{ad}T_n^{cc}-T_n^{ad}T_0^{cc})+[T_n^{ac},T_1^{cd}] \quad (a\neq d) , \tag{3.8}$$

$$T_{n+1}^{aa}-T_{n+1}^{cc}=\lambda(T_0^{aa}T_n^{cc}-T_n^{aa}T_0^{cc})+[T_n^{ac},T_1^{ca}] , \tag{3.9}$$

where no summation for the repeating indices is taken. Equations (3.8) and (3.9) imply that T_n^{ab} can be determined by iteration for given T_0^{ab} and T_1^{ab}.

Now let us set

$$T_0^{ab}=\sum_{i=1}^{N} I_i^{ab} , \tag{3.10}$$

$$T_1^{ab}=\sum_{i=1}^{N} I_i^{ab}D_i , \tag{3.11}$$

and

$$[I_i^{ab},I_j^{cd}]=\delta_{ij}(\delta_{bc}I_i^{ad}-\delta_{ad}I_i^{cb}) , \tag{3.12}$$

where D_i are operators to be determined. Substituting Eqs. (3.10)–(3.12) into f_1^{1}, we obtain

$$\sum_{i}\sum_{j} I_i^{ab}[D^{i},I_j^{cd}]=0 . \tag{3.13}$$

Further, we assume

$$\sum_{i} I_i^{ab}[D_i,I_j^{cd}]=0, \quad \text{for any } j , \tag{3.14}$$

with which the T_2^{ab} should satisfy

$$\delta_{bc}T_2^{ad}-\delta_{ad}T_2^{cb}=\sum_{i\neq j} I_i^{ab}I_j^{cd}\left\{\lambda\sum_{k,l} I_i^{kl}I_j^{lk}(D_j-D_i)+[D_i,D_j]\right\}+\sum_{i}(\delta_{bc}I_i^{ad}D_i^{2}-\delta_{ad}I_i^{cb}D_i^{2}) . \tag{3.15}$$

A sufficient solution of Eq. (3.15) is

$$T_2^{ab}=\sum_{i} I_i^{ab}D_i^{2} , \tag{3.16}$$

with

$$[D_i,D_j]=\lambda\sum_{a,b} I_j^{ab}I_i^{ba}(D_i-D_j) . \tag{3.17}$$

Thus Eq. (3.11) generates a long-range interaction through Eqs. (3.14) and (3.17). However, so far there is no simple relationship between D_i and I_j^{ab} that should satisfy Eq. (3.14). It is very difficult to determine the general relationship. Fortunately, BGHP [10] have set up the link with the help of projection. Let the permutation groups Σ_1, Σ_2, and Σ_3 be generated by K_{ij}, P_{ij}, and the product $P_{ij}K_{ij}$, respectively, where K_{ij} exchange the positions of particles and P_{ij} exchange the spins as positions i and j. The projection ρ was defined as

$$\rho(ab)=a \quad \text{for } \forall_a\in\Sigma_2, \quad b\in\Sigma_1 , \tag{3.18}$$

i.e., the wave function considered is symmetric. Let I_i^{ab} be the fundamental representations; then

$$P_{ij}=\sum_{a,b} I_i^{ab} I_j^{ba} . \tag{3.19}$$

Suppose that [10]

$$D_i=\rho(\hat{D}_i),\quad D_i\in\Sigma_2,\quad \hat{D}_i\in\Sigma_1 \tag{3.20}$$

and the $\hat{D}_i$ are particlelike operators, i.e.,

$$K_{ij}\hat{D}_i=\hat{D}_j K_{ij},\quad K_{ij}\hat{D}_l=\hat{D}_l K_{ij}\quad (l\neq i,j) . \tag{3.21}$$

Define

$$T_m^{ab}=\sum_i I_i^{ab}\rho(\hat{D}_i^m)\quad (m\geq 0) ; \tag{3.22}$$

then (a)

$$[\hat{D}_j,\hat{D}_i]=\lambda\rho^{-1}(P_{ij}(D_j-D_i))=\lambda(\hat{D}_j-\hat{D}_i)K_{ij} . \tag{3.23}$$

(b) T_m^{ab} satisfy Eq. (3.6), i.e., the RTT relation, Eq. (3.2).

Actually $f_1^n=0$ is easy to check. By using

$$[\hat{D}_i^n,\hat{D}_j^m]=\sum_{k=0}^{n-1} D_i^k[\hat{D}_i,\hat{D}_j^m]\hat{D}_j^{n-k-1}$$
$$=\lambda\sum_{k=0}^{n-1} D_i^k(\hat{D}_i^m-\hat{D}_j^m)\hat{D}_j^{n-k-1}K_{ij} ,$$

we have $f_3^{n,m}=0$. The projection procedure is very important, for it enables us to prove that Eq. (3.6) is satisfied by virtue of Eq. (3.20).

With the expansion equation (3.3) and the projected long-range expansion equation (3.22), the Hamiltonian associated with $T(u)$ is obtained by the expansion of the deformed determinant [10]:

$$\det_q T(u)=\sum_\sigma \epsilon(\sigma)T_{1\sigma_1}(u-(p-1)\lambda) \times T_{2\sigma_2}(u-(p-2)\lambda)\cdots T_{p\sigma_p}(u) . \tag{3.24}$$

A calculation gives

$$\det_q T(u)=1+\frac{\lambda}{u}M+\frac{\lambda}{u^2}\left[\rho\sum_i \hat{D}_i-\frac{\lambda}{2}\sum_{j\neq i}K_{ij})+\frac{\lambda}{2}M(M-1)\right] +\frac{\lambda}{u^3}\rho\left\{\left[\sum_i \hat{D}_i-\frac{\lambda}{2}\sum_{j\neq i}K_{ij}\right]^2+\frac{\lambda^2}{12}\sum_{i\neq j\neq k\neq i}K_{ij}K_{jk}+\lambda(M-1)\sum_i\left[\hat{D}_i-\frac{\lambda}{2}\sum_{j\neq i}K_{ij}\right] +\frac{\lambda^2}{6}M(M-1)(M-2)+\frac{\lambda^2}{4}M(M-1)\right\}+\cdots . \tag{3.25}$$

One takes the Hamiltonian as

$$H=\tfrac{1}{2}\rho\left\{\left[\sum_i \hat{D}_i-\frac{\lambda}{2}\sum_{i\neq j}K_{ij}\right]^2+\frac{\lambda^2}{12}\sum_{i\neq j\neq k\neq i}K_{ij}K_{jk}\right\} . \tag{3.26}$$

Therefore we define the Hamiltonian that has the Yangian symmetry given by Eqs. (3.22), (3.12), and (3.17). For a comparison to the known models, we list the expressions for $\hat{D}_i$ satisfying Eq. (3.23):

$$\hat{D}_i=p_i+\frac{\lambda}{2}\sum_{i\neq j}[\mathrm{sgn}(x_i-x_j)+1]K_{ij},\quad \lambda=2il ,$$

$$H=\tfrac{1}{2}\sum_i p_i^2+\tfrac{1}{2}\sum_{i\neq j} l(l-P_{ij})\delta(x_i-x_j) . \tag{3.27}$$

$$\hat{D}_i=p_i+\sum_{i\neq j} l\{i\cot[a(x_i-x_j)]+1\}K_{ij},\quad \lambda=2l ,$$
$$H=\tfrac{1}{2}\sum_i p_i^2+\tfrac{1}{2}\sum_{i\neq j}\frac{l(l-aP_{ij})}{\sin^2[a(x_i-x_j)]} . \tag{3.28}$$

$$\hat{D}_i=p_i+il\sum_{i\neq j}\{\coth[a(x_i-x_j)]P_{ij}^+ +\tanh[a(x_i-x_j)]P_{ij}^-+1\}K_{ij} ,\lambda=2il ,$$
$$H=\tfrac{1}{2}\sum_i p_i^2+\tfrac{1}{2}\sum_{i\neq j} l(l-aP_{ij})\left[\frac{P_{ij}^+}{\sinh^2[a(x_i-x_j)]}-\frac{P_{ij}^-}{\cosh^2[a(x_i-x_j)]}\right] . \tag{3.29}$$

Equations (3.27) and (3.28) were given in Ref. [5] and Eq. (3.28) was studied in Ref. [10]. Equation (3.29) is the generalization of the SR model.

An alternative description of the transfer matrix was given by BGHP [10]. Define

$$\bar{D}_i=\hat{D}_i-\lambda\sum_{i<j}K_{ij} ; \tag{3.30}$$

then

$$[\bar{D}_i,\bar{D}_j]=0 , \tag{3.31}$$

$$[K_{ij},\bar{D}_k]=0\quad (k\neq i,j) , \tag{3.32}$$

$$K_{ij}\bar{D}_i-\bar{D}_j K_{ij}=\lambda . \tag{3.33}$$

It was proved that

$$\bar{T}_i(u)=1+\lambda\frac{P_{0i}}{u-\bar{D}_i},\quad \bar{T}(u)=\prod_i \bar{T}_i(u)\ \text{and}\ \rho(\bar{T}(u)) \tag{3.34}$$

all satisfy the RTT relation.

The deformed determinant of $\overline{T}(u)$ was defined by

$$\det_q \overline{T}(u)=\frac{\Delta_M(u+\lambda)}{\Delta_m(u)}, \quad \Delta_M(u)=\prod_{i=1}^{M}(u-\overline{D}_i) . \tag{3.35}$$

It was proved that

$$\rho(\det_q \overline{T}(u))=\det_q[T(u)] . \tag{3.36}$$

To contain the model, Eq. (2.23), we define $\overline{D}_i$ related to the $\overline{\pi}_i$ given by Eq. (2.16) as

$$\overline{D}_i=\overline{\pi}_i-\beta\sum_{j<i}P_{ij}^{+}K_{ij} , \tag{3.37}$$

which satisfies Eqs. (3.31), (3.32), (3.34), etc. So we can put the models, Eqs. (2.7) and (2.23), into the Yang-Baxter system.

In conclusion of this section we have shown the consistence between the Yangian symmetry and the integrability of the Polychronakos approach for long-range-interaction models, and have given an interpretation of the SR model and the Yan model from the point of view of the YB system.

IV. REFLECTION ALGEBRA

The associativity of the RTT relation, Eq. (3.2), is the Yang-Baxter equation (YBE) [13,14] ($\check{R}(u)=PR(u)$):

$$\check{R}_{12}(u)\check{R}_{23}(u+v)\check{R}_{12}(v)=\check{R}_{23}(v)\check{R}_{12}(u+v)\check{R}_{23}(u) , \tag{4.1}$$

where the subscripts indicate the spaces, namely, $1\to0$, $2\to0'$, and $3\to0''$ in comparison with Eq. (3.2). It is well known that for a given $\check{R}(u)$ satisfying Eq. (4.1) there is allowed a corresponding reflection operator $K(u)$ determined by [16]

$$\check{R}(u-v)K_1(u)\check{R}(u+v)K_1(v)=K_1(v)\check{R}(u+v)K_1(u)\check{R}(u-v) , \tag{4.2}$$

where $K_1(u)=K(u)\otimes 1$. Equation (4.2) possesses the following remarkable properties [16]:

(i) Suppose that $K_\pm(u)$ are c-number solutions of Eq. (4.2). Then

$$\tilde{K}_\pm(u)=T(u)K_\pm(u)T^{-1}(-u) \tag{4.3}$$

also satisfy Eq. (4.2) .

(ii) Define

$$t(u)=\mathrm{tr}[K_+(u+\lambda)T(u)K_-(u)T^{-1}(-u)] ; \tag{4.4}$$

then

$$[t(u),t(v)]=0 , \tag{4.5}$$

i.e., $t(u)$ forms a commuting family. In order to solve $K(u)$ in Eq. (4.2) we make the following expansion:

$$K_0(u)=\sum_{a,b}\sum_{n}X_{ab}^{0}K_{ab}^{(n)}u^{-n} . \tag{4.6}$$

Substituting Eq. (4.6) into Eq. (4.2) one obtains, after calculations,

$$\delta_{bc}[K^{(n)},K^{(m)}]_{ad}+\delta_{ac}\sum_{e}(K_{be}^{(n+1)}K_{ed}^{(m)}+K_{be}^{(n)}K_{ed}^{(m+1)})+\delta_{bd}\sum_{e}(K_{ae}^{(m+1)}K_{ec}^{(n)}-K_{ae}^{(n)}K_{ec}^{(m+1)})$$
$$+[K_{bc}^{(n+2)},K_{ad}^{(m)}]-[K_{bc}^{(n)},K_{ad}^{(m+2)}]+[K_{ac}^{(n+1)}K_{bd}^{(m)}-K_{ac}^{(m)}K_{bd}^{(n+1)}+K_{ac}^{(n)}K_{bd}^{(m+1)}-K_{ac}^{(m+1)}K_{bd}^{(n)}]=0 . \tag{4.7}$$

It follows that

$$[K_{ab}^{(0)},K_{cd}^{(m)}]=0 . \tag{4.8}$$

Suppose $K_{ab}^{(0)}=\delta_{ab}$; the iteration relation reads

$$\delta_{bd}K_{ac}^{(m+2)}-\delta_{ac}K_{bd}^{(m+2)}=\tfrac{1}{2}\{\delta_{ac}[K^{(2)},K^{(m)}]_{bd}-\delta_{bd}[K^{(1)},K^{(m+1)}]_{ac}+K_{ac}^{(2)}K_{bd}^{(m)}-K_{ac}^{(m)}K_{bd}^{(2)}+K_{ac}^{(1)}K_{bd}^{(m+1)}-K_{ac}^{(m+1)}K_{bd}^{(1)}$$
$$+\delta_{bc}[K^{(1)},K^{(m)}]_{bd}+[K_{bc}^{(3)},K_{ad}^{(m)}]\} \quad (m>1) . \tag{4.9}$$

Equation (4.10) tells us that $K^{(m)}$ can be found if $K^{(1)}$, $K^{(2)}$, and $K^{(3)}$ are given properly.

Now let us consider the simplest case where $K(u)$ is a 2×2 matrix given by Eq. (4.12) (see below). Denote

$$T(u)=\begin{bmatrix}T_{11}(u) & T_{12}(u)\\ T_{21}(u) & T_{22}(u)\end{bmatrix} ; \tag{4.10}$$

then

$$T^{-1}(u)=[\det_q T(u)]^{-1}\times\begin{bmatrix}T_{22}(u-\lambda) & -T_{12}(u-\lambda)\\ -T_{21}(u-\lambda) & T_{11}(u-\lambda)\end{bmatrix} . \tag{4.11}$$

Since $\det_q T(u)$ commutes with $T_{ab}(v)$, one does not care about the common factor appearing in Eq. (4.11). We consider the simplest case when $K_\pm=1$ and denote

$$K(u)=T(u)T^{-1}(-u) . \tag{4.12}$$

Now let us see what happens for the long-range-interaction model where $T(u)$ is given by Eq. (3.22). Note that

$$K_{11}(u)=T_{11}(u)T_{22}(-u-\lambda)-T_{12}(u)T_{21}(-u-\lambda),$$
$$K_{12}(u)=T_{12}(u)T_{11}(-u-\lambda)-T_{11}(u)T_{12}(-u-\lambda),$$
$$K_{21}(u)=T_{21}(u)T_{22}(-u-\lambda)-T_{22}(u)T_{21}(-u-\lambda),$$
$$K_{22}(u)=T_{22}(u)T_{11}(-u-\lambda)-T_{21}(u)T_{12}(-u-\lambda). \quad (4.13)$$

The $T_{ab}(u)$ in Eq. (4.13) can be expanded in the terms of Eqs. (3.3) and (3.22), which give the $T_{ab}(u)$:

$$T_{ab}(u)=\delta_{ab}+\lambda\sum_i I_i^{ba}d_i(u), \quad (4.14)$$

where $d_i(u)=\rho(1/(u-\hat{D}_i))$. Substituting Eq. (4.14) into Eq. (4.13) we find

$$K_{11}(u)=1+\lambda\sum_i[I_i^{11}d_i(u)+I_i^{22}d_i(-u-\lambda)-\lambda I_i^{22}d_i(u)d_i(-u-\lambda)]+\lambda^2\sum_{i\neq j}(I_i^{11}I_j^{22}-I_i^{21}I_j^{12})d_i(u)d_j(u-\lambda),$$
$$K_{12}(u)=\lambda\sum_I I_i^{21}[d_i(u)-d_i(-u-\lambda)+\lambda d_i(u)d_i(-u-\lambda)]+\lambda^2\sum_{i\neq j}(I_i^{21}I_j^{11}-I_j^{21}I_i^{11})d_i(u)d_j(-u-\lambda),$$
$$K_{21}(u)=\lambda\sum_I I_i^{12}[d_i(u)-d_i(-u-\lambda)+\lambda d_i(u)d_i(-u-\lambda)]+\lambda^2\sum_{i\neq j}(I_i^{12}I_j^{22}-I_j^{12}I_i^{22})d_i(u)d_j(-u-\lambda),$$
$$K_{22}(u)=1+\lambda\sum_i[I_i^{22}d_i(u)+I_i^{11}d_i(-u-\lambda)-\lambda I_i^{11}d_i(u)d_i(-u-\lambda)]+\lambda^2\sum_{i\neq j}(I_i^{22}I_j^{11}-I_i^{12}I_j^{21})d_i(u)d_j(u-\lambda) \quad (4.15)$$

and

$$t(u)=K_{11}(u)+K_{22}(u)$$
$$=2+\frac{\lambda}{u^2}\left[2\sum_i D_i+\lambda\sum_{i\neq J}P_{ij}+C_1\right]+\frac{\lambda^2}{u^3}\left[\sum_i D_i+\lambda\sum_{i\neq j}P_{ij}+C_2\right]$$
$$+\frac{\lambda}{u^4}\sum_i\left\{2\rho\left[\hat{D}_i+\frac{\lambda}{2}\right]^3-2(N-1)\lambda D_i-(N-1)\lambda^2 D_i\right.$$
$$\left.+\lambda\sum_{j\neq i}\rho(\hat{D}_i\hat{D}_j)+2\lambda\sum_{j\neq i}P_{ij}\rho(\hat{D}_i^2)+\lambda^2\sum_{j\neq i}P_{ij}D_i+\lambda^3\sum_{j\neq i}P_{ij}-\lambda\sum_{j\neq i}P_{ij}\rho(\hat{D}_i\hat{D}_j)\right\}+O(u^{-4}), \quad (4.16)$$

where C_1 and C_2 are constants. Obviously the second term commutes with the third one on the right-hand side of Eq. (4.17). Here we would like to emphasize that the $t(u)$ does not generate conserved quantities.

The physical meaning of Eq. (4.16) for the long-range-interaction models is not clear yet. It requires more knowledge in this area to be explored. What we would like to say is that the simplest form of reflection matrix $K(u)$ for long-range-interaction models can really be calculated. Substituting a variety of forms of $\hat{D}_i$ given in Sec. III, the reflection matrix $K(u)$ can explicitly be expressed by the interactions.

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